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1. INTRODUCTION

The present work is concerned with the determination, by theoretical analysis, of the elastic moduli of a randomly heterogeneous material.

It is assumed that the material may be adequately described by an elastic, homogeneous and isotropic matrix the moduli of which are known, in which particles of another elastic homogeneous and isotropic material are imbedded, the moduli of which are also given. Assuming furthermore that the volume concentration of particles is uniform and that the material may accordingly be regarded as quasi-homogeneous the problem is to find expressions for the effective elastic moduli of this heterogeneous material.

Numerous papers on the determination of bulk properties of heterogeneous materials have been published. The first of these was by Einstein (1906, 1911) in which the viscosity of a suspension was determined, assuming that it may be described by rigid spheres suspended in a viscous fluid and that the volume concentration is so small that particles do not interact. The case of dilute concentration, assuming that the particles are spherical, has been solved for a variety of materials: Liquid droplets in another liquid, Taylor (1932); Elastic particles in viscous fluid; Froehlich and Sack (1946); Empty holes in elastic solid, Mackenzie (1950); Rigid particles in elastic solid, Hashin (1955); Elastic particles in another elastic material, Eshelby (1957), Hashin (1958); Viscous liquid inclusions in elastic solid, Oldroyd (1956). Methods of analysis for small concentration have been recently applied by Budiansky, Hashin and Sanders (1960) to the

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theoretical determination of the early plastic behavior of polycrystalline materials.

In the case of small volume concentration the fundamental assumption is that particles do not interact. The effective moduli can then be determined in a way outlined below (Section 2).

The problem of finite concentration is much more complicated. There is at present little hope of rigorously solving the problem of a medium in which there are many interacting inclusions. The difficulty of this kind of problem is well illustrated by Sternberg's and Sadowsky's (1952) solution of the axisymmetric problem of the theory of elasticity containing only two spherical cavities. [See also Miyamoto (1956).]

Most of the work which has been done up to now in the field of finite concentration has been concerned with the extension of Einstein's formula, previously mentioned, to higher concentrations. This has resulted in a large number of different formulae which are sometimes in direct contradiction. Up to now no formula has been derived which is theoretically well founded and fits experimental results for the whole range of volume concentration. A review of these investigations is not within the scope of this paper. Summaries of these may be found in Frisch and Simha (1956) and Reiner (1958).

Instead of an attempt to find an exact formula for the moduli of a heterogeneous body this work will be concerned with the construction of approximate upper and lower bounds for the moduli. Obviously such bounds are of practical value only if they are close together. It will be found that the expressions for the bulk modulus bounds coincide. In the case of the shear modulus they are mostly close together, as is illustrated by a specific case which has been solved numerically in this paper. Also a simple expression

is derived, which is always smaller than the upper bound and larger than the lower one and can thus be used as a good approximation to the shear modulus whenever the bounds are close.

Apparently the first bounds on an elastic modulus of a heterogeneous material have been given in a recent paper by Paul (1959). Bounds for the Young's modulus of a heterogeneous material were obtained by using the variational theorems of the theory of elasticity and taking as an admissible stress system (see Section 2) the same simple tension in matrix and particles and a simple tension deformation for an admissible displacement field. While these bounds have the advantage of being exact they are far apart because of the simple admissible stress and displacement fields chosen. Thus for example for the same material, for which a numerical solution has been given in this paper, [experimental results for this case have been obtained by Nishimatsu and Gurland (1959) and will be given below] Paul found that for a concentration of 50 per cent, the upper bound for Young's modulus was 43 per cent higher than the lower bound.

In the following general expressions for the elastic moduli of a quasi-homogeneous heterogeneous material will be developed, involving only the stresses or strains inside the particles. This is done by considering the change in strain energy in a loaded homogeneous body due to the insertion of non-homogeneities.

It is then shown that bounds for the moduli can be obtained by suitable choice of admissible str[illegible] fields. In order to evaluate the bounds approximately two geometrical approximations are made. It is assumed, as in small concentration theory, that the particles are spherical and moreover that the action of the whole heterogeneous material on any one

particle is transmitted through a spherical shell which lies wholly in the matrix.

It is believed that this approximation is a close one and comparison of theoretical and experimental results strengthens this belief.

2. GENERAL THEORY

Let an elastic homogeneous and isotropic body of arbitrary shape (Fig. 1a) be subjected to surface tractions which are associated with a homogeneous stress tensor

$$T_i^{(o)} = \sigma_{ij}^{(o)} n_j \tag{2.1}$$

$i, j = 1, 2, 3$ (a repeated subscript denotes summation)

where $T_i^{(o)}$ are the components of the surface stress vector

$\sigma_{ij}^{(o)}$ constant stresses

n_j components of the normal to the outer surface

The strain energy U_o stored in the body is then given by either of the formulae

$$U_o = \frac{1}{2} \int_{(S)} T_i^{(o)} u_i^{(o)} \, dS \tag{2.2}$$

$$U_o = \frac{1}{2} \int_{(V)} \sigma_{ij}^{(o)} \epsilon_{ij}^{(o)} \, dV \tag{2.3}$$

where the $\sigma_{ij}^{(o)}$ and the strains $\epsilon_{ij}^{(o)}$ are connected by Hooke's law

$$\sigma_{ij}^{(o)} = \lambda_m \epsilon^{(o)} \delta_{ij} + 2 G_m \epsilon_{ij}^{(o)} \tag{2.4}$$

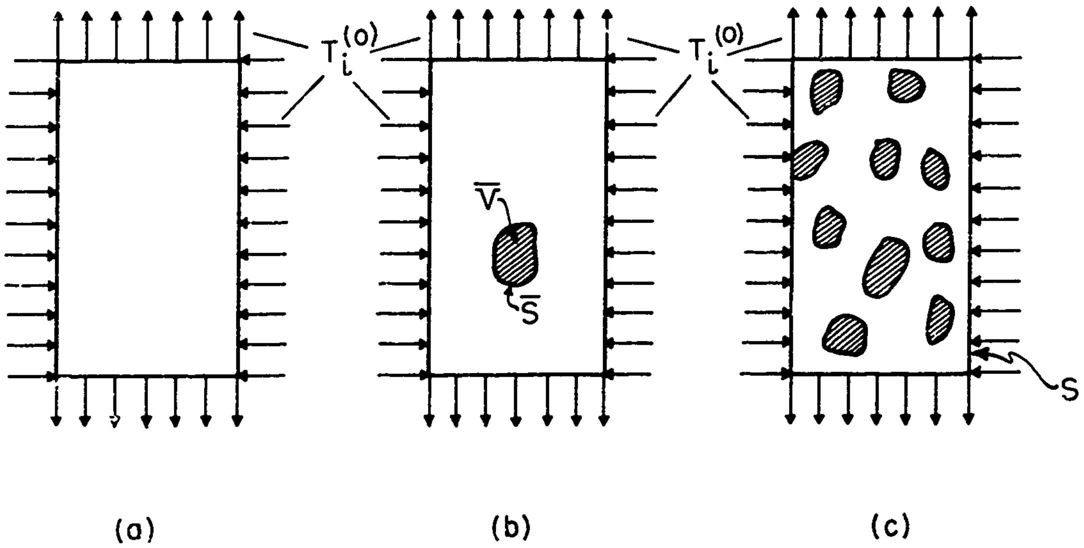


FIG. I HOMOGENEOUS AND NONHOMOGENEOUS BODIES UNDER SAME SURFACE TRACTIONS

In (2.4) λ_m, G_m are the Lamé constant and shear modulus respectively, of the material. $\epsilon^{(o)}$ is given by

$$\epsilon^{(o)} = \epsilon^{(o)}_{kk} = \epsilon^{(o)}_{11} + \epsilon^{(o)}_{22} + \epsilon^{(o)}_{33}$$

and δ_{ij} is the Kronecker delta.

As the strains are homogeneous the elastic displacements $u_i^{(o)}$ are given by

$$u_i^{(o)} = \epsilon^{(o)}_{ij} x_j \tag{2.5}$$

If any stress and strain tensors are split into isotropic and deviatoric parts as follows:

$$\begin{aligned} \sigma_{ij} &= \frac{\sigma}{3} \delta_{ij} + s_{ij} \\ \epsilon_{ij} &= \frac{\epsilon}{3} \delta_{ij} + e_{ij} \end{aligned} \tag{2.6}$$

Hooke's law assumes the form

$$\sigma = 3K\epsilon \qquad s_{ij} = 2Ge_{ij} \tag{2.7}$$

where $K = \lambda + \frac{2}{3} G$ is the bulk modulus and $\sigma = \sigma_{kk}$.

The strain energy density may then be written in the forms:

$$w = \frac{1}{2} \sigma_{ij} \epsilon_{ij} = \frac{1}{2} (K\epsilon^2 + 2Ge_{ij} e_{ij}) \tag{2.8}$$

$$= \frac{1}{2} \left(\frac{\sigma^2}{9K} + \frac{s_{ij} s_{ij}}{2G} \right) \tag{2.9}$$

If part of the body is replaced by an inclusion of another material, the elastic moduli of which are K_p, λ_p, G_p (Fig. 1b) and the same surface tractions are applied, then it has been shown by Eshelby (1951, 1956) that the difference in elastic energy stored in the two cases is given by

$$\delta U^{(\sigma)} = U^{(\sigma)} - U_o^{(\sigma)} = \frac{1}{2} \int_{(\bar{S})} (T_i^{(o)} \bar{u}_i - \bar{T}_i u_i^{(o)}) \, dS \qquad (2.10)$$

where $\bar{S}$ is the surface of the inclusion, $\bar{T}_i$ and $\bar{u}_i$ are the stress and displacement vectors on the surface of the inclusion.

Formula (2.10) is general; it is valid for any inclusion shape and any boundary tractions.

An alternative useful form for (2.10) has been given by Eshelby, this is:

$$\delta U^{(\sigma)} = \frac{1}{2} \int_{(\bar{V})} [(K_m - K_p) \epsilon^{(o)} \epsilon + 2 (G_m - G_p) e_{ij}^{(o)} e_{ij}] \, dV \qquad (2.11)$$

This may be rewritten in terms of stresses by using (2.7)

$$\delta U^{(\sigma)} = \frac{1}{2} \int_{(\bar{V})} \left[\frac{K_m - K_p}{9 K_m K_p} \sigma^{(o)} \sigma + \frac{G_m - G_p}{2 G_m G_p} s_{ij}^{(o)} s_{ij} \right] dV \qquad (2.12)$$

where $\bar{V}$ is the volume of the inclusion. Again (2.11) is a special case of Eshelby's result which is valid for a nonisotropic and nonhomogeneous inclusion. When instead of surface tractions, surface displacements are prescribed formulae (2.10), (2.11) and (2.12) change in sign. If δU when displacements are prescribed, is denoted by $\delta U^{(\epsilon)}$ then Eshelby has shown that:

$$\delta U^{(\epsilon)} = - \frac{1}{2} \int_{(\bar{S})} (T_i^{(o)} \bar{u}_i - \bar{T}_i u_i^{(o)}) \, dS \qquad (2.13)$$

The derivation of (2.10), (2.11) and (2.12) remains unchanged when instead of one inclusion an arbitrary number N of them is introduced into the elastic body and surface tractions are held fixed. (Fig. 1c)

Then the contribution of the n^{th} inclusion is:

$$\delta U_n^{(\sigma)} = \frac{1}{2} \int_{(\overline{S}_n)} (T_i^{(o)} \overline{u}_i^{(n)} - \overline{T}_i^{(n)} u_i^{(o)}) \, dS \tag{2.14}$$

and

$$\delta U^{(\sigma)} = \sum_{n=1}^{n=N} U_n^{(\sigma)} \tag{2.15}$$

Analogous results hold for (2.11), (2.12) and (2.13). In the following, treatment of the case where tractions are prescribed will be called the stress approach, when displacements are prescribed - the displacement approach.

The bulk and shear modulus of an elastic body containing a large number of inclusions of another material will now be given in terms of energy expressions. It is first assumed that the $T_i^{(o)}$ are equivalent to isotropic tension

$$T_i^{(o)} = \frac{1}{3} \sigma^{(o)} n_i \tag{2.16}$$

so

$$\sigma_{ij}^{(o)} = \frac{1}{3} \sigma^{(o)} \delta_{ij} \quad ; \qquad s_{ij}^{(o)} = 0 \tag{2.17}$$

Then from (2.12), for the n^{th} inclusion

$$\delta U_n^{(\sigma)} = \frac{1}{2} \frac{K_m - K_p}{9 K_m K_p} \sigma^{(o)} \int_{(\overline{V}_n)} \sigma^{(n)} \, dV \tag{2.18}$$

When instead of isotropic stress, isotropic radial displacement $u_i^{(o)} = \frac{\epsilon^{(o)}}{3} x_i$ is prescribed on the boundary, then from (2.10), (2.11) and (2.13), and proceeding in an analogous way

$$\delta U_n^{(\epsilon)} = -\frac{1}{2} (K_m - K_p) \epsilon^{(o)} \int_{(\bar{V}_n)} \epsilon^{(n)} \, dV \tag{2.19}$$

For the homogeneous body without inclusions

$$U_o^{(\sigma)} = \frac{1}{2} \frac{\sigma^{(o)^2}}{9 K_m} V \tag{2.20}$$

If it is assumed that the heterogeneous material is quasi-homogeneous with bulk modulus K^*,

$$U^{(\sigma)} = \frac{1}{2} \frac{\sigma^{(o)^2}}{9 K^*} V \tag{2.21}$$

where (2.21) may be regarded as a definition of the "effective" modulus. Then,

$$U^{(\sigma)} = U_o^{(\sigma)} + \sum_{n=1}^{n=N} \delta U_n^{(\sigma)} = U_o^{(\sigma)} + \delta U^{(\sigma)} \tag{2.22}$$

which on using (2.20) and (2.21) may be written as:

$$\frac{1}{K^*} = \frac{1}{K_m} + \frac{\delta U^{(\sigma)}}{\frac{\sigma^{(o)^2}}{18} V} \tag{2.23}$$

where $\delta U^{(\sigma)}$ is given by (2.22) and (2.18).

It should be borne in mind that (2.23) is valid only when the actual $\sigma^{(n)}$ are used in (2.18).

Using now the displacement approach the following expressions are obtained:

$$U_o^{(\epsilon)} = \frac{1}{2} K_m \epsilon^{(o)^2} V \tag{2.24}$$

$$U^{(\epsilon)} = \frac{1}{2} K^* \epsilon^{(o)^2} V \tag{2.25}$$

$$K^* = K_m + \frac{\delta U^{(\epsilon)}}{\frac{1}{2} \epsilon^{(o)^2} V} \tag{2.26}$$

Analogous formulae can be obtained for the shear modulus G^*. It is convenient to apply the stress vector field

$$\begin{aligned} T_1^{(o)} &= \tau n_2 \\ T_2^{(o)} &= \tau n_1 \\ T_3^{(o)} &= 0 \end{aligned} \tag{2.27}$$

which is equivalent to the state of stress,

$$\begin{aligned} &\sigma^{(o)} = 0 \qquad\qquad s_{12}^{(o)} = s_{21}^{(o)} = \tau \\ &s_{11}^{(o)} = s_{22}^{(o)} = s_{33}^{(o)} = s_{31}^{(o)} = s_{13}^{(o)} = 0 \end{aligned} \tag{2.28}$$

which is a pure shear.

Then from (2.12):

$$\delta U_n^{(\sigma)} = \frac{G_m - G_p}{G_m G_p} \tau \int_{(\overline{V}_n)} s_{12}^{(n)} dV \tag{2.29}$$

Also,

$$U_o^{(\sigma)} = \frac{1}{2 G_m} \tau^2 V \tag{2.30}$$

$$U^{(\sigma)} = \frac{1}{2 G^*} \tau^2 V \tag{2.31}$$

and using these expressions in (2.22)

$$\frac{1}{G^*} = \frac{1}{G_m} + \frac{\delta U^{(\sigma)}}{\frac{1}{2}\tau^2 V} \tag{2.32}$$

Proceeding in an analogous way when homogeneous shear displacements are prescribed on the boundary, given by

$$u_1^{(o)} = \frac{\gamma}{2} x_2 \quad ; \quad u_2^{(o)} = \frac{\gamma}{2} x_1 \quad ; \quad u_3^{(o)} = 0 \tag{2.33}$$

the following results are obtained:

$$U_o^{(\epsilon)} = \frac{1}{2} G_m \gamma^2 V \tag{2.34}$$

$$U^{(\epsilon)} = \frac{1}{2} G^* \gamma^2 V \tag{2.35}$$

$$G^* = G_m + \frac{\delta U^{(\epsilon)}}{\frac{1}{2}\gamma^2 V} \tag{2.36}$$

where

$$\delta U^{(\epsilon)} = \sum_{n=1}^{n=N} \delta U_n^{(\epsilon)}$$

$$\delta U_n^{(\epsilon)} = \frac{1}{4} (G_m - G_p) \gamma \int_{(\overline{V}_n)} \gamma^{(n)} dV \tag{2.37}$$

The expressions for the effective moduli depend on the actual stresses or strains in the inclusions. In the case of small volume concentration, when it is assumed that there is no interaction, these may be determined by assuming that each inclusion is in an infinite medium where the stresses or strains at infinity are the ones applied to the surface of the body. Solving the boundary value problem, the expressions for the case of small concentration

are easily obtained. This procedure is clearly impossible for finite concentration. On the other hand <u>bounds</u> on the moduli may be obtained by using the variational theorems of the theory of elasticity.

The theorems of minimum complementary energy and of minimum potential energy will be here used in their following special forms:

(a) When tractions are prescribed over the entire surface of an elastic body and the body forces vanish, then of all sets of stresses, satisfying the equilibrium conditions and boundary conditions, the actual state of stress minimizes the strain energy $U^{(\sigma)}$.

(b) When displacements are prescribed over the entire surface of an elastic body and the body forces vanish, then of all sets of displacements, satisfying the boundary conditions, the actual displacements minimize the strain energy $U^{(\epsilon)}$. [c.f., e.g. Sokolnikoff -- Mathematical Theory of Elasticity (1956).] Thus for the special cases mentioned, both theorems reduce to a minimum principle for the strain energy.

In what follows a field of stress or displacements belonging to the sets described in (a) or (b) respectively will be called admissible.

If in the stress approach any admissible state of stress is chosen, the strain energy associated with it may be computed from (2.12) and (2.22). If the expression $\delta U^{(\sigma)}$ thus determined is denoted by $\delta \widetilde{U}^{(\sigma)}$ and $U^{(\sigma)}$ is the actual strain energy, then from (2.22) and principle (a):

$$U^{(\sigma)} \leq U_0^{(\sigma)} + \delta \widetilde{U}^{(\sigma)} \tag{2.38}$$

If now in the displacement approach an admissible field of displacements is chosen, the strain energy is computed from (2.11) and (2.22) and the expression $\delta U^{(\epsilon)}$ is denoted by $\delta \widetilde{U}^{(\epsilon)}$ -- then from principle (b)

$$U^{(\epsilon)} \le U_0^{(\epsilon)} + \delta \tilde{U}^{(\epsilon)} \tag{2.39}$$

The bounds on the bulk modulus are obtained in the following way: An admissible state of stress is chosen, so that on the boundary S of the body shown in Fig. 1c (2.16) is satisfied, and $\delta \tilde{U}^{(\sigma)}$ is determined.

From (2.38), (2.20) and (2.21)

$$\frac{1}{K^*} \le \frac{1}{K_m} + \frac{\tilde{U}^{(\sigma)}}{\frac{\sigma^{(0)^2}}{18} V}$$

which may be rewritten in the form:

$$K^* \geqslant \frac{K_m}{1 + \frac{\delta \tilde{U}^{(\sigma)}}{\frac{\sigma^{(0)^2}}{18 K_m} V}} = K_1^* \tag{2.40}$$

Accordingly K_1^* is a <u>lower bound</u> for K^*.

The same procedure is now applied to the displacement approach. Displacements are chosen so that on the boundary

$$u_i^{(0)} = \frac{\epsilon^{(0)}}{3} x_i \tag{2.41}$$

Using (2.24), (2.25) and (2.39) the following inequality is obtained

$$K^* \leqslant K_m + \frac{\delta \tilde{U}^{(\epsilon)}}{\frac{1}{2} \epsilon^{(0)^2} V} = K_2^* \tag{2.42}$$

Then K_2^* is an upper bound for K^*. So from (2.40) and (2.42)

$$K_1^* \leq K^* \leq K_2^* \tag{2.43}$$

Exactly the same method may be used for the shear modulus and it is then found that

$$G_1^* \leq G^* \leq G_2^* \tag{2.44}$$

where

$$G_1^* = \frac{G_m}{1 + \dfrac{\delta \tilde{U}^{(\sigma)}}{\dfrac{\tau^2}{2\,G_m} V}} \tag{2.45}$$

$$G_2^* = G_m + \frac{\delta \tilde{U}^{(\epsilon)}}{\frac{1}{2} \gamma^2 V} \tag{2.46}$$

It should be remembered that $\delta \tilde{U}^{(\sigma)}$ in (2.40) and (2.45) are different quantities. The same applies to $\delta \tilde{U}^{(\epsilon)}$ in (2.42) and (2.46).

The theoretical treatment presented up to this point has been exact, within the framework of the linear theory of elasticity. In the following some reasonable assumptions will have to be made.

Consider a body containing a large number of inclusions which are uniformly distributed in it. (Fig. 2) The body is very large in comparison to an inclusion. The volume concentration of inclusions is then defined by

$$c = \frac{\sum_{n=1}^{n=N} \overline{V}_n}{V} \tag{2.47}$$

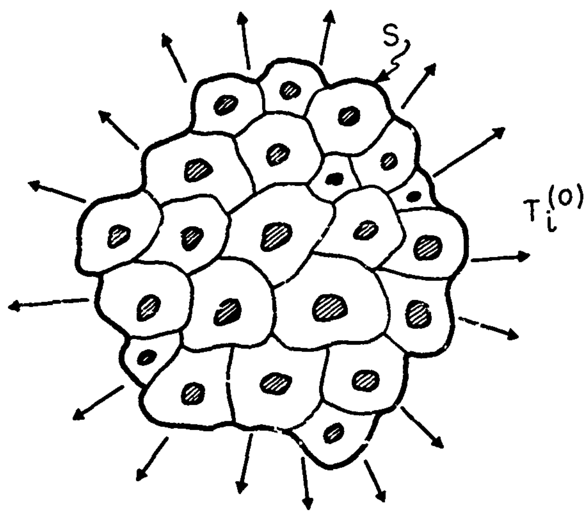


FIG. 2 HETEROGENEOUS MATERIAL DIVIDED INTO COMPOSITE ELEMENTS

where $\overline{V}_n$ is the volume of the n^{th} inclusion and V the volume of the whole body. By uniform concentration the following is understood: when choosing an arbitrary, not too small, volume element of the body the fractional volume of inclusions in it is expected to deviate from c only by a small amount.

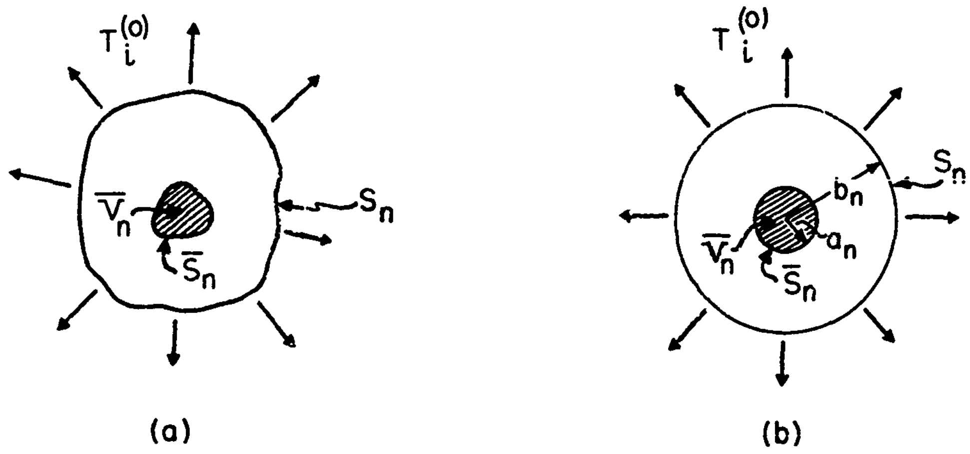
A system of tractions $T_i^{(o)}$ derived from homogeneous stresses is now applied to the surface of the body. Each inclusion is then imagined to be surrounded by a surface S_n, enclosing a volume V_n, so that for <u>every</u> inclusion

$$\frac{\overline{V}_n}{V_n} = c \tag{2.48}$$

It is possible to construct these surfaces S_n in an infinity of ways and it will be assumed that this has been done so as to approach spherical surfaces as nearly as possible. In the following any element, containing an inclusion and enclosed by S_n will be called - a composite element. (Fig. 3a)

In order to choose a state of stress $\tilde{\sigma}_{ij}$ which satisfies equilibrium and boundary conditions it will be assumed that every surface S_n of a composite element is loaded by the tractions $T_i^{(o)}$ acting on S. If then the boundary value problems for the composite elements are solved, the stresses thus determined fulfil the necessary conditions for an admissible stress system $\tilde{\sigma}_{ij}$. In order to solve the problem it is assumed that the composite element may be approximated by two concentric spheres so that volumes are preserved (Fig. 3b). Such an element will be called - a spherical composite element.

For the displacement approach it is assumed that the surface displacements of S are linear and that the surface displacements of each S_n are of the same linear form. When the corresponding boundary value problem for



COMPOSITE ELEMENT

SPHERICAL COMPOSITE ELEMENT

FIGURE 3

the composite element is solved the resulting field of displacements will be admissible.

3. APPROXIMATE BOUNDS FOR THE BULK MODULUS

(a) Lower bound by stress approach.

The spherical composite element is given by a sphere of radius b_n concentric with a spherical particle of radius a_n. By definition

$$\frac{\overline{V}_n}{V_n} = \frac{a_n^3}{b_n^3} = \rho^3 \tag{3.1}$$

where

$$\rho^3 = c \tag{3.2}$$

The outer surface S of the heterogeneous body is loaded by a constant radial stress

$$\sigma_{rr}^{(o)} = \frac{\sigma^{(o)}}{3} = \tilde{\sigma} \tag{3.3}$$

Then the same stress is applied to the spherical surface $r = b$. (In the following the subscript n will be dropped.)

The elastic moduli of the inclusion are K_p, G_p. Those of the spherical shell - K_m, G_m.

The problem is one of radial symmetry the general solution of which is [(Love), p. 142]:

$$\left.\begin{aligned} u_r^{(p)} &= A_p r + \frac{B_p}{r^2} \\ \sigma_{rr}^{(p)} &= 3\,K_p\,A_p - 4\,G_p\,\frac{B_p}{r^3} \end{aligned}\right\} \text{For the inside of the inclusion.} \quad (3.4)$$

$$\left.\begin{aligned} u_r^{(m)} &= A_m r + \frac{B_m}{r^2} \\ \sigma_{rr}^{(m)} &= 3\,K_m\,A_m - 4\,G_m\,\frac{B_m}{r^3} \end{aligned}\right\} \text{For the spherical shell.} \quad (3.5)$$

The four unknown constants may easily be determined from the four boundary conditions:

$$u_r^{(p)} = 0 \qquad (r = 0) \qquad (3.6)$$

$$\left.\begin{aligned} u_r^{(p)} &= u_r^{(m)} \\ \sigma_{rr}^{(p)} &= \sigma_{rr}^{(m)} \end{aligned}\right\} \qquad (r = a) \qquad (3.7)$$

$$\sigma_{rr}^{(p)} = \tilde{\sigma} \qquad (r = b) \qquad (3.8)$$

From the preceding analysis it follows that only the stresses inside the inclusion are of interest. It is found that:

$$A_p = \frac{\tilde{\sigma}}{3}\;\frac{4\,G_m + 3\,K_m}{K_m\,(4\,G_m + 3\,K_p) - 4\,G_m\,(K_m - K_p)\,\dfrac{a^3}{b^3}} \qquad (3.9)$$

$$B_p = 0$$

Then

$$\sigma_{rr}^{(p)} = 3\,K_p\,A_p \tag{3.10}$$

From (2.18), (3.1), (3.2), (3.9) and (3.10)

$$\delta\tilde{U}_n^{(\sigma)} = \frac{1}{2}\,\tilde{\sigma}^2\,\frac{K_m - K_p}{K_m}\cdot\frac{4\,G_m + 3\,K_m}{K_m\,(4G_m + 3K_p) - 4G_m\,(K_m - K_p)\,c}\,\bar{V}_n \tag{3.11}$$

From (2.40) and (2.47),

$$K_1^* = \frac{K_m}{1 + \dfrac{(K_m - K_p)\,(4\,G_m + 3\,K_m)c}{K_m\,(4\,G_m + 3\,K_p) - 4\,G_m\,(K_m - K_p)\,c}} \tag{3.12}$$

(b) Upper bound by displacement approach.

It is here assumed that the boundary displacement of S is purely radial

$$u_i^{(o)} = \frac{\epsilon^{(o)}}{3}\,x_i = \tilde{\epsilon}\,x_i \tag{3.13}$$

Accordingly the displacement applied to the boundary of the spherical composite element is

$$u_r^{(o)} = \tilde{\epsilon}\,r \tag{3.14}$$

The boundary value problem is solved exactly as before, the only difference being in that equation (3.8) has to be replaced by (3.14) for $r = b$. It is then found:

$$A_p = \tilde{\epsilon} \; \frac{4\,G_m + 3\,K_m}{4\,G_m + 3\,K_p + 3\,(K_m - K_p)\,\dfrac{a^3}{b^3}} \tag{3.15}$$

$$B_p = 0 \tag{3.16}$$

From (2.19), (3.1) and (3.15),

$$\delta U_n^{(\epsilon)} = \frac{9}{2}\,\tilde{\epsilon}^2\,(K_m - K_p)\,\frac{4\,G_m + 3\,K_m}{4G_m + 3\,K_p + 3\,(K_m - K_p)\,c}\;\bar{V}_n \tag{3.17}$$

and from (2.42) and (2.47)

$$K_2^* = K_m + (K_p - K_m)\,\frac{(4\,G_m + 3\,K_m)\,c}{4\,G_m + 3\,K_p + 3\,(K_m - K_p)\,c} \tag{3.18}$$

Comparison of K_1^* given by (3.12) and K_2^* given by (3.18) shows that the <u>upper and lower bounds coincide</u>.

$$K_1^* = K_2^* = K^* \tag{3.19}$$

As the bounds are approximate, this does not necessarily show that the expression for K^* is exact.

The expression (3.18) can be rewritten in the form

$$\frac{K^*}{K_m} = 1 + \frac{3\,(1 - \nu_m)\left(\dfrac{K_p}{K_m} - 1\right)c}{2\,(1 - 2\,\nu_m) + (1 + \nu_m)\left[\dfrac{K_p}{K_m} - \left(\dfrac{K_p}{K_m} - 1\right)c\right]} \tag{3.20}$$

It can also be rearranged into the symmetric form

$$c = \frac{\left(f + \dfrac{K_p}{K_m}\right)\left(1 - \dfrac{K^*}{K_m}\right)}{\left(f + \dfrac{K^*}{K_m}\right)\left(1 - \dfrac{K_p}{K_m}\right)} \tag{3.21}$$

where f is given by

$$f = \frac{4\,G_m}{3\,K_m} = \frac{2(1 - 2\,\nu_m)}{1 + \nu_m}$$

and ν_m is the Poisson's ratio of the matrix. Generalization to the case when the particles are of different kinds is obvious. Let there be k kinds of particles imbedded in the matrix. Let $K_p^{(i)}$ be the bulk modulus of the i^{th} particle kind and c_i its volume concentration. Then (3.20) is generalized to:

$$\frac{K^*}{K_m} = 1 + 3(1 - \nu_m) \sum_{i=1}^{i=k} \frac{\left(\frac{K_p^{(i)}}{K_m} - 1\right) c_i}{2(1 - 2\,\nu_m) + (1 + \nu_m)\left[\frac{K_p^{(i)}}{K_m} - \left(\frac{K_p^{(i)}}{K_m} - 1\right) c\right]} \tag{3.22}$$

where

$$c = \sum_{i=1}^{i=k} c_i$$

4. APPROXIMATE BOUNDS FOR THE SHEAR MODULUS

The method is essentially the same as for the bulk modulus. A homogeneous shear stress or homogeneous shear displacement is applied to the boundary S and also to the spherical surface S_n of the spherical composite element. Because of the absence of radial symmetry the boundary value problem to be solved is much more difficult than that of the preceding section. Its formulation is as follows:

Solve the two systems of field equations of the theory of elasticity:

$$\left.\begin{aligned} (\lambda_p + G_p)\, \epsilon^{(p)}_{,i} + G_p \nabla^2 u^{(p)}_i &= 0 \\ \epsilon^{(p)} &= u^{(p)}_{i,i} \end{aligned}\right\} \qquad (4.1)$$

$$\left.\begin{aligned} (\lambda_m + G_m)\, \epsilon^{(m)}_{,i} + G_m \nabla^2 u^{(m)}_i &= 0 \\ \epsilon^{(m)} &= u^{(m)}_{i,i} \end{aligned}\right\} \qquad (4.2)$$

where: $i = 1, 2, 3$, a comma denotes partial differentiation, a subscript or superscript p denotes the inclusion, a subscript or superscript m denotes the shell, $\nabla^2 \equiv \dfrac{\partial^2}{\partial x_i \, \partial x_i}$ denotes Laplace's operator.

The boundary conditions are:

$$u^{(p)}_i = 0 \qquad (r = 0) \qquad (4.3)$$

$$\left.\begin{aligned} u^{(p)}_i &= u^{(m)}_i \qquad (4.4) \\ T^{(p)}_i &= T^{(m)}_i \qquad (4.5) \end{aligned}\right\} \quad (r = a)$$

$$\left.\begin{aligned} T^{(m)}_1 &= \tau n_2 \\ T^{(m)}_2 &= \tau n_1 \\ T^{(m)}_3 &= 0 \end{aligned}\right\} \quad (r = b) \qquad (4.6)$$

Equations (4.6) express the condition that pure shear, $\sigma_{12} = \tau$, is applied to the boundary in the stress approach. The direction cosines n_i are given, for a spherical surface, by

$$n_i = \frac{x_i}{r} \qquad (4.7)$$

In the displacement approach a homogeneous shear displacement is applied to the boundary. Then (4.6) is replaced by:

$$\left.\begin{aligned} u_1^{(m)} &= \frac{1}{2}\gamma x_2 \\ u_2^{(m)} &= \frac{1}{2}\gamma x_1 \\ u_3^{(m)} &= 0 \end{aligned}\right\} \qquad (r = b) \tag{4.8}$$

An exact closed solution in terms of solid spherical harmonics is given in the appendix to this paper. Proceeding in a way analogous to that of section 3, the following expressions for the approximate bounds for the shear modulus are found (see appendix A)

$$G_1^* = \frac{G_m}{1 + (1 - \eta)\, y_1^{(\sigma)}\, c} \tag{4.9}$$

$$G_2^* = G_m\,[1 + (\eta - 1)\, y_1^{(\epsilon)}\, c] \tag{4.10}$$

in which

$$\eta = \frac{G_p}{G_m}$$

and $y_1^{(\sigma)}$ and $y_1^{(\epsilon)}$ are defined by equations (A-28) and (A-31) in appendix A.

So according to (2.44)

$$G_1^* \leq G^* \leq G_2^*$$

As for the bulk modulus the results can be easily generalized to the case of different kinds of particles. Using the following notation:

$G_p^{(i)}$ shear modulus of i^{th} kind of particles

$$\eta_i = \frac{G_p^{(i)}}{G_m}$$

$$\rho^3 = c = \sum_{i=1}^{i=k} c_i$$

where k as before, is the number of different kinds of particles, equations (4.9) and (4.10) are now replaced by,

$$G_1^* = G_m \sum_{i=1}^{i=k} \frac{1}{1 + (1 - \eta_i)\, y_{1(i)}^{(\sigma)}\, c_i} \tag{4.11}$$

$$G_2^* = G_m \left[1 + \sum_{i=1}^{i=k} (\eta_i - 1)\, y_{1(i)}^{(\epsilon)}\, c_i \right] \tag{4.12}$$

The $y_{1(i)}^{(\sigma)}$ and $y_{1(i)}^{(\epsilon)}$ are determined from (A-28) and (A-31) for the values $G_p^{(i)}$ and $\nu_p^{(i)}$. The value of ρ is not affected and is as given above.

The results for the shear modulus are much more complicated than the simple expression for the bulk modulus obtained in section 3. Whenever the bounds for the shear modulus are close together a formula which gives values which lie between them can be regarded as a good approximation. A simple formula of this kind is derived in appendix B. The result is:

$$\frac{\overline{G^*}}{G_m} = 1 + \frac{15\,(1 - \nu_m)\left(\frac{G_p}{G_m} - 1\right) c}{7 - 5\nu_m + 2\,(4 - 5\nu_m)\left[\frac{G_p}{G_m} - \left(\frac{G_p}{G_m} - 1\right) c\right]} \tag{4.13}$$

When there are k different kinds of particles

$$\frac{\overline{G}^*}{G_m} = 1 + 15(1 - \nu_m) \sum_{i=1}^{i=k} \frac{\left(\frac{G_p^{(i)}}{G_m} - 1\right) c_i}{7 - 5\nu_m + 2(4 - 5\nu_m)\frac{G_p^{(i)}}{G_m} - 2(4 - 5\nu_m)\left(\frac{G_p^{(i)}}{G_m} - 1\right) c} \tag{4.14}$$

It is shown in appendix B that

$$G_1^* \leq \overline{G}^* \leq G_2^* \tag{4.15}$$

In view of the numerical calculations and comparison with experiments which will be given in the next section it is necessary to determine what bounds on Young's modulus are given by the bounds on the shear modulus. It is easily proved that an upper or lower bound on the shear modulus gives the corresponding bound for Young's modulus. Assuming that G_1 and G_2 are lower and upper bounds, i.e.

$$G_2 > G_1 \tag{4.16}$$

Then

$$E_1 = \frac{9 K G_1}{3 K + G_1}$$

$$E_2 = \frac{9 K G_2}{3 K + G_2}$$

for the bounds for K coincide. Then,

$$\frac{E_2}{E_1} = \frac{1 + \frac{3K}{G_1}}{1 + \frac{3K}{G_2}} > 1$$

from inequality (4.16). So

$$E_2 > E_1 \tag{4.17}$$

It is easily proved, in a similar way, from the expression of Poisson's ratio by the bulk and shear moduli, that the upper bound to the shear modulus corresponds to a lower bound on Poisson's ratio and vice versa.

5. EXPRESSIONS FOR THE MODULI FOR VERY SMALL AND VERY LARGE CONCENTRATIONS

A small concentration formula for a modulus of a heterogeneous material is one that is valid when squares and higher powers of the volume concentration of the particles can be neglected. Thus for very small c

$$\frac{K^*}{K_m} = 1 + \alpha c \tag{5.1}$$

$$\frac{G^*}{G_m} = 1 + \beta c \tag{5.2}$$

The numbers α and β may be interpreted as the slopes of the curves "modulus-concentration" at $c = 0$.

Similarly large concentration formulae can be defined by

$$\frac{K^*}{K_p} = 1 + \alpha' c' \tag{5.3}$$

$$\frac{G^*}{G_p} = 1 + \beta' c' \tag{5.4}$$

in which

$$c' = 1 - c \tag{5.5}$$

and c is very close to unity.

The value of α is easily found from (3.20) when the term containing c, in the denominator, is neglected. The following expression is obtained.

$$\frac{K^*}{K_m} = 1 - \frac{3(1-\nu_m)\left(1-\frac{K_p}{K_m}\right)}{2(1-2\nu_m)+(1+\nu_m)\frac{K_p}{K_m}}\, c \tag{5.6}$$

For the shear modulus equations (A-28) and (A-31) have to be solved for the case where $\rho^3 = c$ is very small. The upper and lower bound expressions for small concentration are then found from (A-35) and (A-36). It is found that both expressions are the same and reduce to:

$$\frac{G^*}{G_m} = 1 - \frac{15(1-\nu_m)\left(1-\frac{G_p}{G_m}\right)}{7-5\nu_m+2(4-5\nu_m)\frac{G_p}{G_m}}\, c \tag{5.7}$$

Thus the upper and lower bound have the same slope at $c = 0$. Expressions (5.6) and (5.7) have already been given, in the same form by Hashin (1958) and in another form by Eshelby (1957).

The same procedure can be used for very large concentrations. The results are:

$$\frac{K^*}{K_p} = 1 - \frac{\left(1-\frac{K_m}{K_p}\right)\left[2(1-2\nu_m)+(1+\nu_m)\frac{K_p}{K_m}\right]}{3(1-\nu_m)}\, c' \tag{5.8}$$

$$\frac{G^*}{G_p} = 1 - \frac{\left(1 - \frac{G_m}{G_p}\right)\left[7 - 5\nu_m + 2\,(4 - 5\nu_m)\,\frac{G_p}{G_m}\right]}{15\,(1 - \nu_m)}\,c' \qquad (5.9)$$

Again the expressions for the shear modulus are the same; so the upper and lower bound have the same slope also at $c = 1$.

It is worthwhile to note that there is a simple relation between α and α' defined in (5.1), (5.3) and β and β' defined in (5.2) and (5.4). From (5.6) and (5.8)

$$\alpha\alpha' = \left(1 - \frac{K_p}{K_m}\right)\left(1 - \frac{K_m}{K_p}\right) \qquad (5.10)$$

From (5.7) and (5.9)

$$\beta\beta' = \left(1 - \frac{G_p}{G_m}\right)\left(1 - \frac{G_m}{G_p}\right) \qquad (5.11)$$

Formulae for Young's modulus for small and large concentration can now be easily derived. The relation between Young's modulus E and the bulk and shear modulus is given by:

$$E = \frac{9KG}{3K + G} \qquad (5.12)$$

Introducing (5.1) and (5.2) into (5.12) and linearizing with respect to c gives

$$\frac{E^*}{E_m} = 1 + \frac{3K_m\beta + G_m\alpha}{3K_m + G_m}\,c \qquad (5.13)$$

Similarly for large concentration, using (5.3) and (5.4),

$$\frac{E^*}{E_p} = 1 + \frac{3K_p\beta' + G_p\alpha'}{3K_p + G_p}\,c' \qquad (5.14)$$

The quantities α, β, α' and β' are given in (5.6) - (5.9).

6. COMPARISON OF THEORETICAL AND EXPERIMENTAL RESULTS

The theoretical results obtained above will now be used for a numerical comparison with experimental measurements.

A suitable example is given by Tungsten Carbide-Cobalt alloys (WC-Co) the micro-structure and mechanical behavior of which have been extensively studied. The most recent experimental work on this alloy is given in papers by Gurland (1959) and Nishimatsu and Gurland (1959).

The alloy prepared consisted of WC particles imbedded in a matrix of Co. The experimental results of chief interest for the present work are measurements of Young's modulus of the alloy for varying volume concentrations of one phase relative to the other. These results are given in Table No. 1. Another quantity of interest is the <u>contiguity</u> which is defined as the average fraction of surface area shared by a grain of WC with all neighboring grains of the same phase. It has been assumed in the theoretical analysis that the contiguity is zero for all concentrations. This can clearly not be expected to hold for an actual alloy. Table No. 1 contains the measured contiguities. The volume concentration c refers to the WC phase.

The values of the contiguities show that there is a definite preference for the Co phase to be classified as the matrix. This is also shown by photographs of the microstructure of the alloy, at different concentrations, which are given in the above cited papers.

In accordance with the preceding analysis the moduli of the WC will be given the subscript p and those of the Co the subscript m. Then in the present case

$$E_m = 30 \times 10^6 \text{ psi} \qquad E_p = 102 \times 10^6 \text{ psi}$$

$$\nu_m = 0.30 \qquad \nu_p = 0.22$$

Using the formulae

$$K = \frac{E}{3(1 - 2\nu)} \qquad (6.1)$$

$$G = \frac{E}{2(1 + \nu)} \qquad (6.2)$$

the values of the bulk and shear moduli for the two phases are:

$K_m = 25.0 \times 10^6$ psi $\qquad K_p = 60.7 \times 10^6$ psi

$G_m = 11.5 \times 10^6$ psi $\qquad G_p = 41.8 \times 10^6$ psi

The variation with volume concentration of the bulk modulus K^*, of the bounds for the shear modulus G_1^* and G_2^* and of $\overline{G}^*$, have been determined by the method given in the theoretical part below. Young's modulus and Poisson's ratio have then been calculated from the expressions:

$$E = \frac{9KG}{3K + G} \qquad (6.3)$$

$$\nu = \frac{3K - 2G}{2(3K + G)} \qquad (6.4)$$

In (6.3) and (6.4), K^* given by (3.20) is used for K. The bounds on E^* and ν^* are obtained when introducing the bounds G_1^* and G_2^* for G. The quantities $\overline{E}^*$ and $\overline{\nu}^*$, which are obtained when using $\overline{G}^*$ (given by (4.13)) for G, may be regarded as approximate values for the Young's modulus and the Poisson's ratio of the heterogeneous material, whenever the bounds are close. The results are given in Table No. 2 below in form of ratios to the matrix moduli.

Figure 4 shows the calculated variation of the bulk modulus of the alloy. Figure 5 gives the bounds for the shear modulus and the variation of $\overline{G}^*$, Fig. 6 - the bounds for Poisson's ratio and $\overline{\nu}^*$, Fig. 7 - the bounds for Young's modulus and $\overline{E}^*$. In the variations of the moduli results of formulae for small and large concentration, given in Section 5 are shown as straight lines tangent to the curves at concentrations 0 and 1.0.

It appears from the figures that the bounds are quite close together and are themselves good approximations to the values of the moduli. The experimental results given in Table 1 are plotted in Fig. 7. This shows that the experimental points closely follow the curves predicted by theory.

7. DISCUSSION AND CONCLUSIONS

It has been shown that approximations to bounds for the elastic moduli of composite materials can be determined by use of the variational theorems of the theory of elasticity. For a specific example of an alloy for which experimental data are available it was found that the theoretical results are close to those found by experiment.

The mathematical expressions for the bounds for the shear modulus depend in a rather complex manner on the ratio between the shear moduli of the two materials and their Poisson's ratios. It is therefore difficult to give a general criterion for the closeness of the bounds.

A possible procedure, which is applicable to specific cases, is the following: The quantity Δ given by,

$$\Delta = \frac{G_2^*}{G_1^*} - 1$$

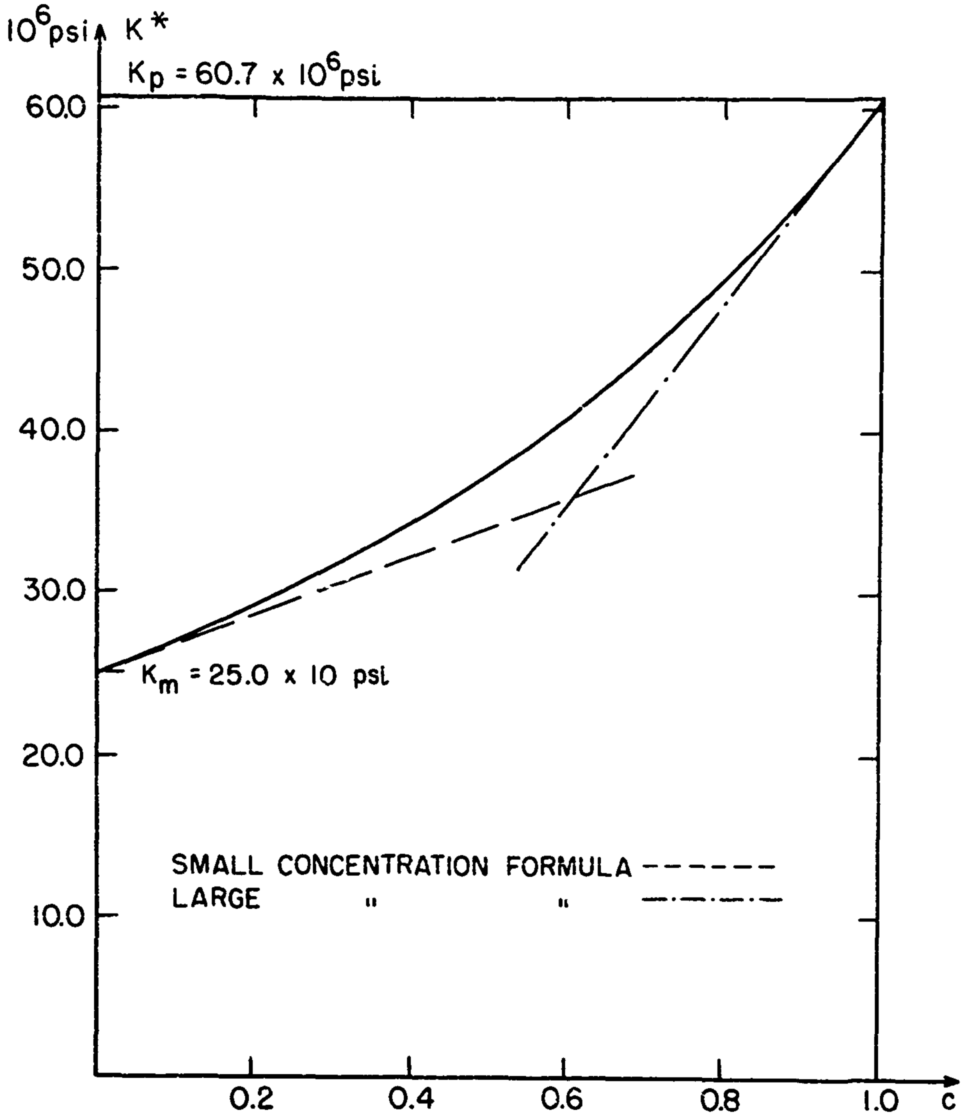


FIG. 4 VARIATION OF BULK MODULUS WITH CONCENTRATION

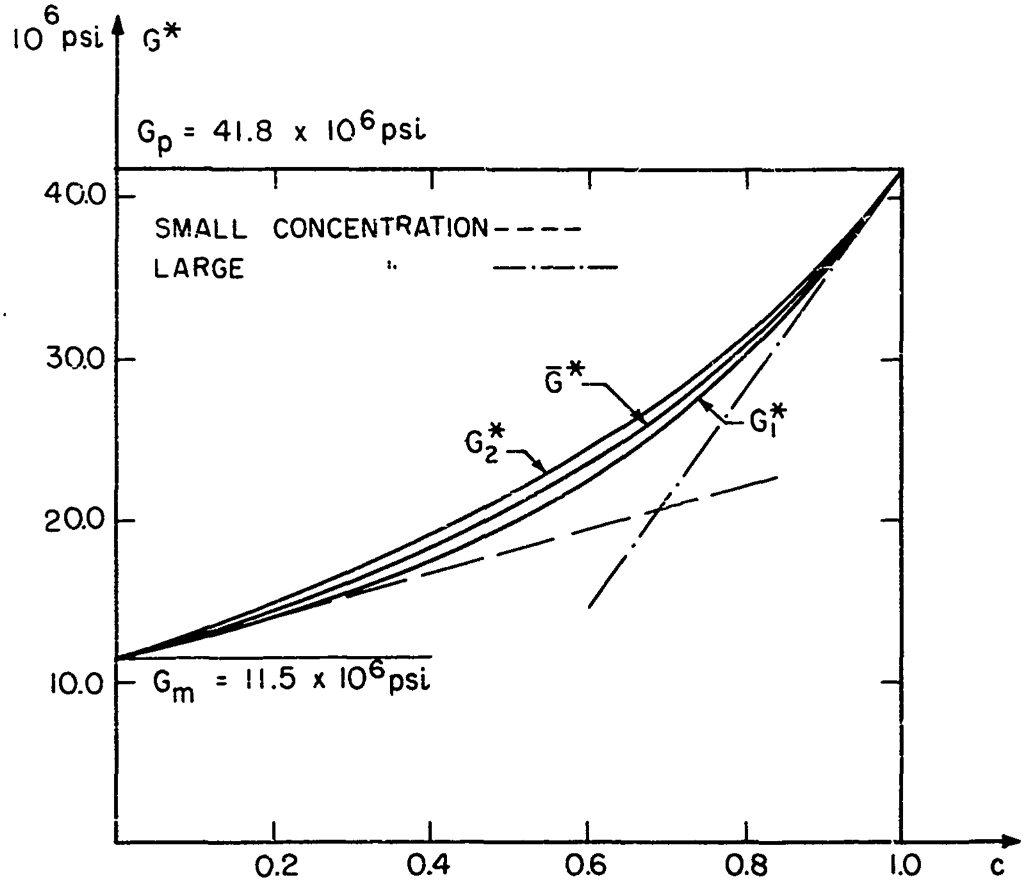


FIG. 5 APPROXIMATE BOUNDS FOR SHEAR MODULUS AND $\bar{G}^*$

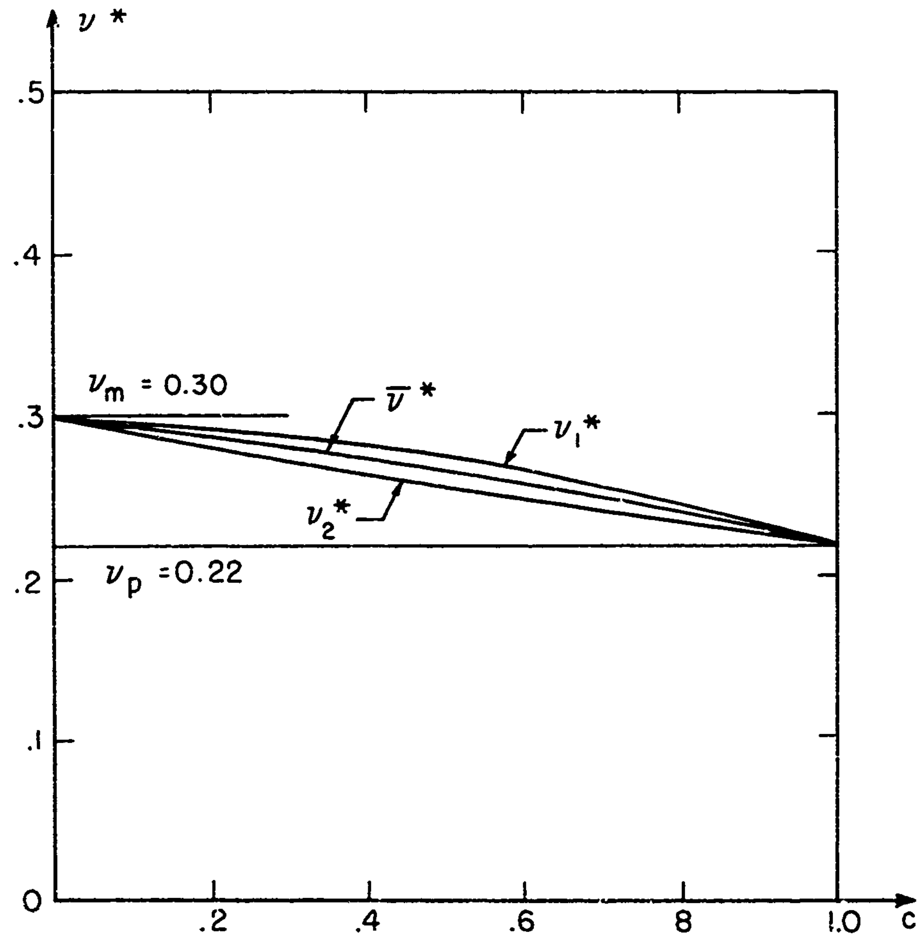


FIG. 6 APPROXIMATE BOUNDS FOR POISSON'S RATIO AND $\overline{\nu}^*$

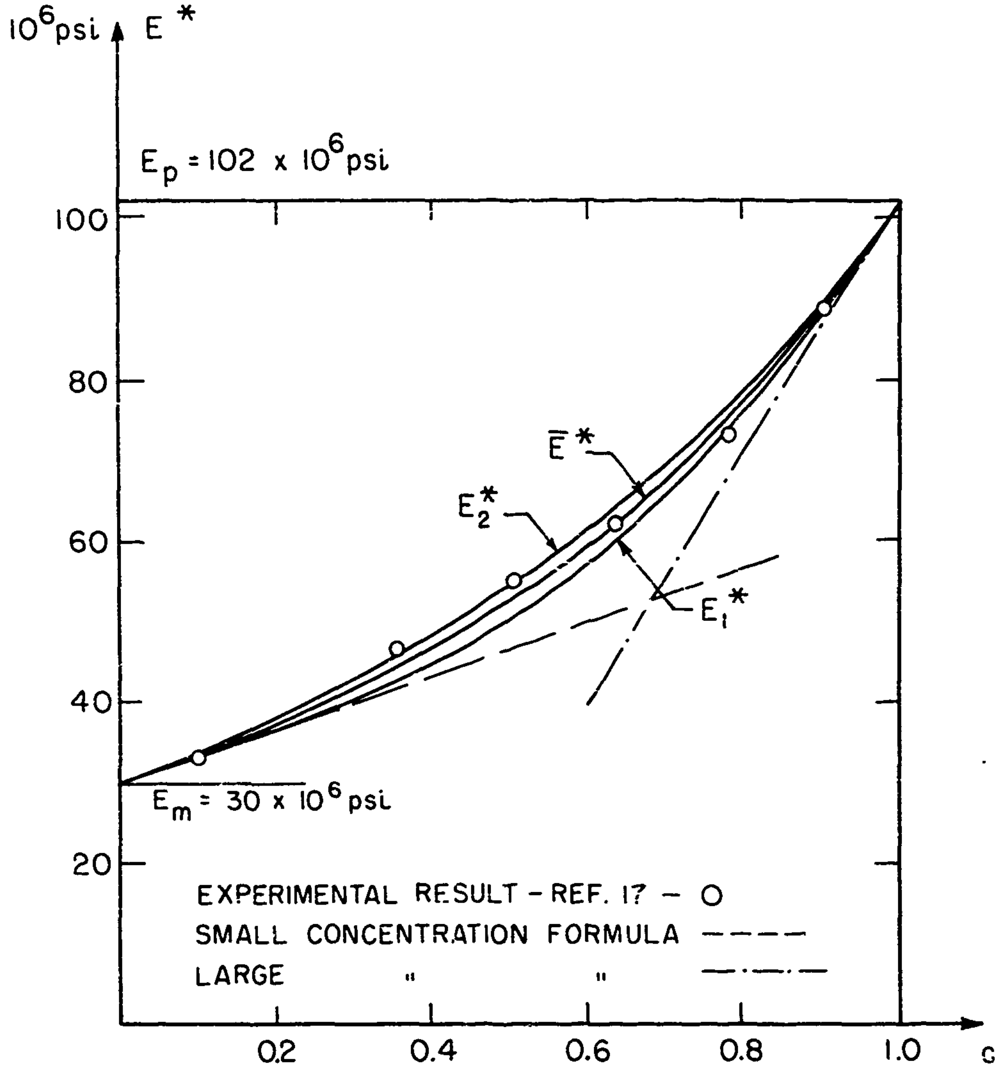


FIG. 7 APPROXIMATE BOUNDS FOR YOUNG'S MODULUS AND $\bar{E}$*

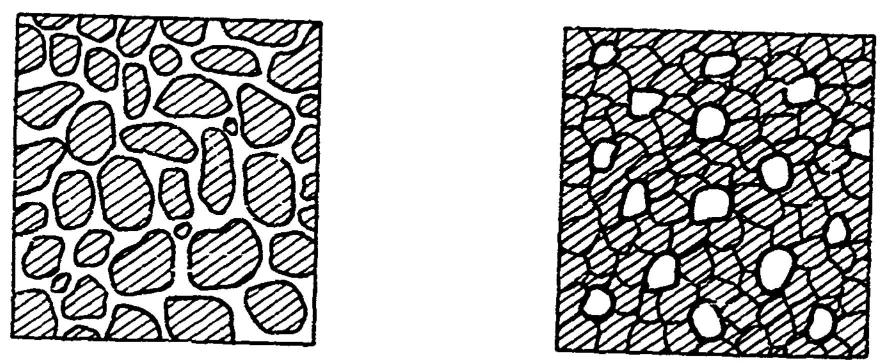
is evaluated numerically at a concentration of 50 per cent. This will give a good estimate of the maximum difference between the bounds. Whenever Δ is small enough, $\bar{G}^*$ given by (4.13) can be used as the shear modulus of the heterogeneous material.

It is to be expected that whenever the difference in moduli between particle and matrix material is not too large, the bounds will be close together. It should be noted that for the WC-Co alloy the ratio η between the shear moduli was 3.62 which is certainly not small. The bounds were nevertheless close together. It is consequently believed that the simple expression $\bar{G}^*$ can be used with good accuracy for many practical cases. A different situation may arise for such extreme cases as rigid particles or empty cavities. This is further discussed below.

An important property of the approximate bounds should be emphasized. At both extremities of the concentration range, $c = 0$ and $c = 1$, the slope is the same for both the upper and lower bounds for any elastic modulus of the heterogeneous material. This provides some additional foundation to small concentration theory and introduces large concentration formulae which, it is believed, are of the same order of accuracy as those for small concentration.

Some consequences of a basic assumption made in this work should be pointed out. It has been assumed, that at any concentration, a particle can always be surrounded by a surface which lies wholly in the matrix. In other words, the matrix always remains connected. A possible geometrical configuration of such a material at high concentration is shown in Fig. 8a.

Another possibility is a heterogeneous material which is an agglomeration of grains of two or more different materials. A concentration of 100 per cent is reached when all grains are replaced by grains of one



(a) (b)

FIG. 8 STRUCTURE OF MATERIAL AT HIGH CONCENTRATION

material. The geometrical configuration of such a material at high concentration is shown in Fig. 8b. In this second case there is no preference of one material over the other. The expressions for the moduli have to be invariant for replacement of one material by another and of c by 1-c.

The analysis given in this paper applies only to the first case and it should be expected that for two such materials, composed of the same constituents, the difference in moduli would increase with increasing volume concentration. This is illuminated by the fact that while small concentration formulae for these two materials are the same, the large concentration formulae are not.

The two materials described should be regarded as extreme theoretical cases. The structure of an actual heterogeneous material will in all probability be somewhere between the two.

The theory developed in this work may be easily applied to the special cases of a body containing empty cavities or a body containing rigid particles. In the first case

$$K_p = 0 \quad ; \quad G_p = 0$$

and in the second

$$K_p \longrightarrow \infty \quad ; \quad G_p \longrightarrow \infty$$

Equations (A-28) and (A-31) are greatly simplified in these two cases and the bounds can be easily expressed in closed form. Numerical calculations show that the bounds so obtained are further apart then for the alloy treated in this work.

The case of cavities is important for the study of porous materials. The solution for rigid particles can by mathematical analogy be applied to the problem of the viscosity of a suspension. As there is complete mathematical analogy between the theory of incompressible elastic media and the theory of Stokes flow (viscous flow when neglecting inertia terms in the Navier-Stokes equations), the results for the shear modulus of an incompressible elastic body containing rigid particles will hold for the coefficient of viscosity of a viscous fluid in slow motion, containing rigid particles. However for both cavities and rigid particles a special difficulty appears - the range of volume concentration will not extend to 100 per cent. When the skeleton of a porous material breaks down the volume of solid of which this skeleton is composed is not negligible in comparison to the volume of the voids. Also the flow of a suspension will cease, i.e. - the coefficient of viscosity becomes infinite, at a concentration of 50 per cent - 60 per cent of particles.

It seems that the method given in this work cannot be applied without modification to these extreme cases.

APPENDIX A

PROBLEM OF SHEARED COMPOSITE SPHERE AND BOUNDS FOR SHEAR MODULUS

In the following a solution to the boundary value problems, set up in section 4, will be constructed.

General solutions for the elastic sphere and the elastic spherical shell have been given by Lord Kelvin (Thomson) and Tait (1879), and Luré (1953) for the general case and by Sternberg, Eubanks and Sadowsky (1957) for the axisymmetric case. It is in principle possible to construct solutions for the spherical inclusion and for the spherical shell, leaving enough arbitrary constants to fit the boundary conditions.

The first two general solutions are very difficult to use owing to the complexity of the general solution for the spherical shell. The axisymmetric solution is more explicit and it can be applied to the present problem by superposition of axisymmetric tension and compression at right angles. A simpler method of solution will be found by proceeding in a less straightforward way.

The problem of a spherical inclusion in an infinite elastic medium, the moduli of which are different from those of the inclusion, when the stress at infinity is uniform (or the displacements are linear) may be solved in a convenient way by application of Kelvin's general solution of the problem of an elastic sphere [(Love) pp. 265-270].

In the case of pure shear in the 1, 2 directions it is found for the region exterior to the inclusion that displacements and accordingly stresses, may be expressed in terms of a solid spherical harmonic of negative integral degree -3,

$$\Psi_{(-3)} = \frac{x_1 x_2}{r^5} \tag{A-1}$$

and its derivatives.

Love (pp. 249-251) has given three special solutions to the field equations of elasticity and it can be shown that the solution of the problem just mentioned may also be found by superposing Love's ω and ϕ solutions. These are in vector form:

<u>Type ω</u>

$$\begin{aligned} \vec{u} &= r^2 \operatorname{grad} \omega_n + \alpha_n \omega_n \vec{r} \\ \alpha_n &= -2 \frac{n\lambda + (3n+1)G}{(n+3)\lambda + (n+5)G} \end{aligned} \tag{A-2}$$

<u>Type ϕ</u>

$$\vec{u} = \operatorname{grad} \phi_n \tag{A-3}$$

For the infinite medium in pure shear the solution can be constructed by putting

$$\omega_n = \phi_n = \Psi_{(-3)}$$

In the problem given by equations (4.1) - (4.8) the exterior region is a <u>finite</u> spherical shell. It is reasonable to expect that the solution can be constructed by using also the solid harmonic of positive integral degree associated with $\Psi_{(-3)}$.

If Ψ_n is a solid harmonic of positive integral degree n then:

$$\Psi_{-n-1} = \frac{\Psi_n}{r^{2n+1}}$$

is a solid harmonic of integral degree $-n-1$. [Compare e.g. MacRobert (1943) p. 74.] So in this case

$$\Psi_2 = x_1 x_2 \tag{A-4}$$

The elastic displacement in the spherical shell can now be written as follows,

$$\vec{u}^{(m)} = \bar{A}_1 \vec{u}^{(1)} + \bar{A}_2 \vec{u}^{(2)} + \bar{A}_3 \vec{u}^{(3)} + \bar{A}_4 \vec{u}^{(4)} \tag{A-5}$$

where,

$$\vec{u}^{(1)} = \text{grad}\ (x_1\ x_2) \tag{A-6}$$

$$\vec{u}^{(2)} = \text{grad}\left(\frac{x_1\ x_2}{r^5}\right) \tag{A-7}$$

$$\vec{u}^{(3)} = r^2\ \text{grad}\ (x_1\ x_2) + \alpha_2^{(m)}\ x_1\ x_2\ \vec{r} \tag{A-8}$$

$$\vec{u}^{(4)} = r^2\ \text{grad}\left(\frac{x_1\ x_2}{r}\right) + \alpha_{(-3)}^{(m)}\ \frac{x_1\ x_2}{r^5}\ \vec{r} \tag{A-9}$$

According to (A-2)

$$\alpha_2^{(m)} = -2\ \frac{2\lambda_m + 7\,G_m}{5\lambda_m + 7\,G_m} \tag{A-10}$$

$$\alpha_{(-3)}^{(m)} = \frac{3\lambda_m + 8\,G_m}{G_m} \tag{A-11}$$

Similarly for the interior of the inclusion

$$\vec{u}^{(p)} = \bar{B}_1 \vec{u}^{(1)} + \bar{B}_3 \vec{u}^{(3)} \tag{A-12}$$

$$\bar{B}_2 = \bar{B}_4 = 0 \tag{A-13}$$

The last equation is derived from the fact that $u^{(p)}$ vanishes for $r = 0$.

For the interior region

$$\vec{u}^{(3)} = r^2 \operatorname{grad} (x_1 x_2) + \alpha_2^{(p)} x_1 x_2 \vec{r} \qquad \text{(A-14)}$$

$$\alpha_2^{(p)} = -2 \frac{2\lambda_p + 7 G_p}{5\lambda_p + 7 G_p} \qquad \text{(A-15)}$$

In the following expressions for the stress vectors on a spherical surface will be needed. These may be determined by differentiation of the displacements and are given for the special type solutions in Love (pp. 250-251). Introducing the nondimensional constants

$$\begin{aligned} A_1 &= \bar{A}_1 \\ A_2 &= \frac{1}{a^5} \bar{A}_2 \\ A_3 &= a^2 \bar{A}_3 \\ A_4 &= \frac{1}{a^3} \bar{A}_4 \\ B_1 &= \bar{B}_1 \\ B_3 &= a^2 \bar{B}_3 \end{aligned} \qquad \text{(A-16)}$$

and using the formula

$$\frac{\lambda}{G} = \frac{2\nu}{1 - 2\nu}$$

where ν is Poisson's ratio - the stress vectors on a sphere of radius r where $a \leq r \leq b$ are given by the following expressions:

$$T_1^{(m)} = \frac{2\,G_m}{r}\left\{\left[A_1 - 4\,A_2 + \frac{7+2\,\nu_m}{7-4\,\nu_m}\,A_3 + \frac{1+\nu_m}{1-2\,\nu_m}\,A_4\right]x_2\right.$$

$$\left. + \left[20\,A_2 - 2\,\frac{7+5\,\nu_m}{7-4\,\nu_m}\,A_3 - \frac{12}{1-2\,\nu_m}\,A_4\right]\frac{x_1^2\,x_2}{r^2}\right\} \tag{A-17}$$

$$T_2^{(m)} = \frac{2\,G_m}{r}\left\{\left[A_1 - 4\,A_2 + \frac{7+2\,\nu_m}{7-4\,\nu_m}\,A_3 + \frac{1+\nu_m}{1-2\,\nu_m}\,A_4\right]x_1\right.$$

$$\left. + \left[20\,A_2 - 2\,\frac{7+5\,\nu_m}{7-4\,\nu_m}\,A_3 - \frac{12}{1-2\,\nu_m}\,A_4\right]\frac{x_1\,x_2^2}{r^2}\right\} \tag{A-18}$$

$$T_3^{(m)} = \frac{2\,G_m}{r}\left[20\,A_2 - 2\,\frac{7+5\,\nu_m}{7-4\,\nu_m}\,A_3 - \frac{12}{1-2\,\nu_m}\,A_4\right]\frac{x_1\,x_2\,x_3}{r^2} \tag{A-19}$$

For the interior of the inclusion $0 \leq r \leq a$

$$T_1^{(p)} = \frac{2\,G_p}{r}\left\{\left[B_1 + \frac{7+2\,\nu_p}{7-4\,\nu_p}\,B_3\right]x_2 - 2\,\frac{7+5\,\nu_p}{7-4\,\nu_p}\,B_3\,\frac{x_1^2\,x_2}{r^2}\right\} \tag{A-20}$$

The remaining components are easily written down by analogy with (A-18) - (A-19).

The expressions for the displacements are: For the shell $a \leq r \leq b$

$$u_1^{(m)} = (A_1 + A_2 + A_3 + A_4)\, x_2 +$$

$$+ \left(-5\, A_2 - 2\, \frac{7 - 10\,\nu_m}{7 - 4\,\nu_m}\, A_3 + \frac{3}{1 - 2\,\nu_m}\, A_4 \right) \frac{x_1^2\, x_2}{r^2} \tag{A-21}$$

$$u_2^{(m)} = (A_1 + A_2 + A_3 + A_4)\, x_1 +$$

$$+ \left(-5\, A_2 - 2\, \frac{7 - 10\,\nu_m}{7 - 4\,\nu_m}\, A_3 + \frac{3}{1 - 2\,\nu_m}\, A_4 \right) \frac{x_1\, x_2^2}{r^2} \tag{A-22}$$

$$u_3^{(m)} = \left(-5\, A_2 - 2\, \frac{7 - 10\,\nu_m}{7 - 4\,\nu_m}\, A_3 + \frac{3}{1 - 2\,\nu_m}\, A_4 \right) \frac{x_1\, x_2\, x_3}{r^2} \tag{A-23}$$

For the interior of the inclusion, $0 \leq r \leq a$:

$$u_1^{(p)} = (B_1 + B_3)\, x_2 - 2\, \frac{7 - 10\,\nu_p}{7 - 4\,\nu_p}\, B_3\, \frac{x_1^2\, x_2}{r^2} \tag{A-24}$$

and the other components follow by analogy.

The two boundary value problems for the elastic composite sphere may now be easily solved. The expressions given for the displacements already satisfy equations (4.1), (4.2). Equation (4.3) is satisfied by the components of $\vec{u}^{(p)}$ given by (A-24). Setting (A-21) equal to (A-24) in accordance with (4.4), (A-17) equal to (A-20) in accordance with (4.5) and equating to zero coefficients of x_2 and $x_1^2\, x_2$ - 4 linear equations for the six unknown coefficients A_1, A_2, A_3, A_4, B_1, B_3 are obtained.

It should be noted that equating of other components of displacement and stress vectors does not give new equations. This follows from the form of the expressions and may easily be understood by the symmetry of the

problem with regard to the bisector plane drawn between the $x_1 x_3$ and $x_2 x_3$ planes.

Two additional equations are obtained for each of the problems, either from (4.6) and (A-17) or from (4.8) and (A-21). The problem is thus reduced to the solution of six linear equations with six unknowns.

For the stress problem the matrix of coefficients is as follows:

	$A_1^{(\sigma)}$	$A_2^{(\sigma)}$	$A_3^{(\sigma)}$	$A_4^{(\sigma)}$	$B_1^{(\sigma)}$	$B_3^{(\sigma)}$	
(1)	1	1	1	1	-1	-1	0
(2)	0	-5	$-2\dfrac{7-10\nu_m}{7-4\nu_m}$	$\dfrac{3}{1-2\nu_m}$	0	$2\dfrac{7-10\nu_p}{7-4\nu_p}$	0
(3)	1	-4	$\dfrac{7+2\nu_m}{7-4\nu_m}$	$\dfrac{1+\nu_m}{1-2\nu_m}$	$-\eta$	$-\eta\dfrac{7+2\nu_p}{7-4\nu_p}$	0
(4)	0	10	$-\dfrac{7+5\nu_m}{7-4\nu_m}$	$-\dfrac{6}{1-2\nu_m}$	0	$\eta\dfrac{7+5\nu_p}{7-4\nu_p}$	0
(5)	0	$10\rho^5$	$-\dfrac{7+5\nu_m}{7-4\nu_m}\dfrac{1}{\rho^2}$	$-\dfrac{6}{1-2\nu_m}\rho^3$	0	0	0
(6)	1	$-4\rho^5$	$\dfrac{7+2\nu_m}{7-4\nu_m}\dfrac{1}{\rho^2}$	$\dfrac{1+\nu_m}{1-2\nu_m}\rho^3$	0	0	$\dfrac{\tau}{2G_m}$

(A-25)

where

$$\eta = \frac{G_p}{G_m} \tag{A-26}$$

and ρ has been defined by (3.1) and (3.2).

For the displacement problem only rows (5) and (6) in (A-25) undergo a change. These are now:

	$A_1^{(\epsilon)}$	$A_2^{(\epsilon)}$	$A_3^{(\epsilon)}$	$A_4^{(\epsilon)}$	$B_1^{(\epsilon)}$	$B_3^{(\epsilon)}$	
(5)	0	$-5\rho^5$	$-2\,\dfrac{7-10\nu_m}{7-4\nu_m}\,\dfrac{1}{\rho^2}$	$\dfrac{3}{1-2\nu_m}\,\rho^3$	0	0	0
(6)	1	ρ^5	$\dfrac{1}{\rho^2}$	ρ^3	0	0	$\dfrac{\delta}{2}$

(A-27)

The system of equations (A-25) may be reduced after various rearrangements and application of rules for evaluation of determinants to a system of two equations in two unknowns:

$$
\begin{aligned}
&y_1^{(\sigma)}\left\{\frac{2}{5}\,\frac{1-\eta}{1-\nu_m}\,(\rho^7-\rho^5)\right\}+y_2^{(\sigma)}\left\{\left[(7-10\nu_p)-(7-10\nu_m)\vartheta\right]4\rho^7-(7+5\nu_m)\vartheta\right\}=0\\
&y_1^{(\sigma)}\left\{\eta+\frac{7-5\nu_m}{15(1-\nu_m)}\,(1-\eta)\,(1-\rho^3)\right\}+y_2^{(\sigma)}\,21\vartheta\left(\frac{1}{\rho^2}-1\right)=1
\end{aligned}
\tag{A-28}
$$

in which,

$$
\left.
\begin{aligned}
y_1^{(\sigma)} &= \frac{2\,G_m}{\tau}\left[B_1^{(\sigma)}+\frac{21}{5\,(7-4\nu_p)}\,B_3^{(\sigma)}\right]\\
y_2^{(\sigma)} &= \frac{2\,G_m}{5\tau}\,B_3^{(\sigma)}
\end{aligned}
\right\}
\tag{A-29}
$$

$$
\vartheta=\frac{4(7-10\nu_p)+\eta\,(7+5\nu_p)}{35\,(1-\nu_m)}
\tag{A-30}
$$

The system of equations associated with (A-27) may be reduced in a similar way to:

$$y_1^{(\epsilon)}\left\{\frac{2}{5}\,\frac{1-\eta}{1-\nu_m}\,(\rho^7-\rho^5)\right\}+y_2^{(\epsilon)}\left\{\left[(7-10\nu_p)-(7-10\nu_m)\vartheta\right]4\rho^7+4(7-10\nu_m)\vartheta\right\}=0$$

$$y_1^{(\epsilon)}\left\{\eta+\frac{7-5\nu_m}{15(1-\nu_m)}(1-\eta)+\frac{2(4-5\nu_m)}{15(1-\nu_m)}(1-\eta)\rho^3\right\}+y_2^{(\epsilon)}\,21\vartheta\left(\frac{1}{\rho^2}-1\right)=1 \tag{A-31}$$

in which

$$\begin{aligned} y_1^{(\epsilon)} &= \frac{2}{\gamma}\left[B_1^{(\epsilon)}+\frac{21}{5(7-4\nu_p)}\,B_3^{(\epsilon)}\right] \\ y_2^{(\epsilon)} &= \frac{2}{5\gamma}\,B_3^{(\epsilon)} \end{aligned} \tag{A-32}$$

and ϑ is given by (A-30).

The quantities $\delta U_n^{(\sigma)}$, $\delta U_n^{(\epsilon)}$ are best determined in this case from equations (2.14) (2.13) using (A-21) and its analogues, and (A-24) and its analogues. The unknown constants B_1, B_3 are determined from (A-28) and (A-29) for the stress approach and from (A-31) and (A-32) for the displacement approach. The $T_i^{(o)}$ and $u_i^{(o)}$ appearing in (2.14) are given by (4.6) and (4.8) respectively. (Because of the theorems that when an elastic body is loaded by constant stress this stress system is found at every point in it; when deformed by linear surface displacements this displacement field is found throughout it.) Carrying out the integration the following expressions are found:

$$\delta U_n^{(\sigma)} = \frac{\tau^2}{2G_m}(1-\eta)\, y_1^{(\sigma)}\, \overline{V}_n \tag{A-33}$$

$$\delta U_n^{(\epsilon)} = \frac{G_m \gamma^2}{2}(\eta - 1)\, y_1^{(\epsilon)}\, \overline{V}_n \tag{A-34}$$

Then from (2.45) and (2.46)

$$G_1^{*} = \frac{G_m}{1 + (1-\eta)\, y_1^{(\sigma)}\, c} \tag{A-35}$$

$$G_2^{*} = G_m\left[1 + (\eta - 1)\, y_1^{(\epsilon)}\, c\right] \tag{A-36}$$

where (2.47) and (2.22) have been used.

The actual expressions for $y_1^{(\sigma)}$ and $y_1^{(\epsilon)}$ are clumsy and it seems best to determine them numerically when numerical values of K_m, G_m and K_p, G_p are given. It should be noted that $y_1^{(\sigma)}$ and $y_1^{(\epsilon)}$ are functions of ρ i.e. of the volume concentration c.

APPENDIX B

APPROXIMATE EXPRESSION FOR SHEAR MODULUS

The expression (4.13) for $\overline{G}^*$ can be derived in the following way. The system of equations (A-28) is written in the form

$$
\begin{aligned}
a_{11}^{(\sigma)} y_1^{(\sigma)} + a_{12}^{(\sigma)} y_2^{(\sigma)} &= 0 \\
a_{21}^{(\sigma)} y_1^{(\sigma)} + a_{22}^{(\sigma)} y_2^{(\sigma)} &= 1
\end{aligned}
\tag{B-1}
$$

And analogously for system (A-37),

$$
\begin{aligned}
a_{11}^{(\epsilon)} y_1^{(\epsilon)} + a_{12}^{(\epsilon)} y_2^{(\epsilon)} &= 0 \\
a_{21}^{(\epsilon)} y_1^{(\epsilon)} + a_{22}^{(\epsilon)} y_2^{(\epsilon)} &= 1
\end{aligned}
\tag{B-2}
$$

Then,

$$
y_1^{(\sigma)} = \frac{1}{a_{21}^{(\sigma)} - \dfrac{a_{11}^{(\sigma)} a_{22}^{(\sigma)}}{a_{12}^{(\sigma)}}} \tag{B-3}
$$

$$
y_1^{(\epsilon)} = \frac{1}{a_{21}^{(\epsilon)} - \dfrac{a_{11}^{(\epsilon)} a_{22}^{(\epsilon)}}{a_{12}^{(\epsilon)}}} \tag{B-4}
$$

The quantities $\overline{y}_1^{(\sigma)}$ and $\overline{y}_1^{(\epsilon)}$ are defined by,

$$
\overline{y}_1^{(\sigma)} = \frac{1}{a_{21}^{(\sigma)}} \tag{B-5}
$$

$$
\overline{y}_1^{(\epsilon)} = \frac{1}{a_{21}^{(\epsilon)}} \tag{B-6}
$$

If $y_1^{(\sigma)}$ and $y_1^{(\epsilon)}$ in equations (4.9) and (4.10) [(A-35) and (A-36) in Appendix A] are replaced by $\overline{y}_1^{(\sigma)}$ and $\overline{y}_1^{(\epsilon)}$, the resulting expressions are:

$$\frac{\overline{G}_1^*}{G_m} = \frac{1}{1 + (1 - \eta)\,\overline{y}_1^{(\sigma)}\, c} \tag{B-7}$$

$$\frac{\overline{G}_2^*}{G_m} = 1 + (\eta - 1)\,\overline{y}_1^{(\epsilon)}\, c \tag{B-8}$$

Introducing the coefficients $a_{21}^{(\sigma)}$ and $a_{21}^{(\epsilon)}$ from (A-28) and (A-31) it turns out that,

$$\overline{G}_1^* = \overline{G}_2^* = \overline{G}^* \tag{B-9}$$

where $\overline{G}^*$ is given by equation (4.13).

It can be proved that for all values of c,

$$G_1^* \leq \overline{G}^* \leq G_2^* \tag{B-10}$$

The proof is somewhat tedious and only its outline will be given here. This is as follows:

(a) The quantities $y_1^{(\sigma)}$ and $y_1^{(\epsilon)}$ are always positive.

From the definition of $\delta U^{(\sigma)}$ and $\delta U^{(\epsilon)}$ it follows that when the moduli of an inclusion are larger than those of the medium $\delta U^{(\sigma)}$ is negative and $\delta U^{(\epsilon)}$ is positive. When the moduli of an inclusion are smaller than those of the medium the converse is true. Applying this to equations (A-33) and (A-34), statement (a) follows.

(b) The following relations hold for the coefficients $a_{ij}^{(\sigma)}$ and $a_{ij}^{(\epsilon)}$.

$$\left.\begin{aligned} a_{11}^{(\sigma)} &= a_{11}^{(\epsilon)} = a_{11} \\ a_{22}^{(\sigma)} &= a_{22}^{(\epsilon)} = a_{22} \\ a_{21}^{(\epsilon)} &- a_{21}^{(\sigma)} = (1-\eta)\, c \\ a_{12}^{(\epsilon)} &- a_{12}^{(\sigma)} = 35(1-\nu_m)\,\vartheta \end{aligned}\right\} \qquad \text{(B-11)}$$

$$\left.\begin{aligned} & a_{11} > 0 && \text{for} \quad \eta > 1 \\ & a_{11} < 0 && \text{for} \quad \eta < 1 \\ & a_{12}^{(\sigma)} < 0 && \\ & a_{21}^{(\sigma)} > 0 && \\ & a_{22} > 0 && \\ & a_{12}^{(\epsilon)} > 0 && \\ & a_{21}^{(\epsilon)} > 0 && \end{aligned}\right\} \qquad \text{(B-12)}$$

(c) The quantities $y_1^{(\sigma)}$ and $y_1^{(\epsilon)}$ satisfy the following inequalities:

$$y_1^{(\sigma)} \geqslant y_1^{(\epsilon)} \qquad \text{for} \quad \eta > 1 \qquad \text{(B-13)}$$

$$y_1^{(\sigma)} \leqslant y_1^{(\epsilon)} \qquad \text{for} \quad \eta < 1 \qquad \text{(B-14)}$$

This follows from (B-3) and (B-4) and the inequalities given in (b).

(d) The approximate bounds satisfy the necessary condition:

$$G_1^* \leq G_2^* \tag{B-15}$$

This follows from (A-35), (A-36), (B-3), (B-4), (B-12), (B-13) and (B-14).

(e) The quantities $\bar{G}_1^*$ and $\bar{G}_2^*$ are related to G_1^* and G_2^* by the following inequalities:

$$G_1^* \leq \bar{G}_1^* \tag{B-16}$$

$$\bar{G}_2^* \leq G_2^* \tag{B-17}$$

This follows by introducing (B-5) and (B-6) into (B-7) and (B-8), (B-3) and (B-4) into (A-35) and (A-36) and then comparing G_1^* with $\bar{G}_1^*$ and G_2^* with $\bar{G}_2^*$, using (B-12).

So according to (B-16), (B-17) and (B-9)-(B-10) is true.

The proof remains essentially unchanged for the case when the particles are of different kinds.

Equation (413) may be rearranged into a symmetric form analogous to (3.21),

$$c = \frac{\left(g + \frac{G_p}{G_m}\right)\left(1 - \frac{G^*}{G_m}\right)}{\left(g + \frac{G^*}{G_m}\right)\left(1 - \frac{G_p}{G_m}\right)} \tag{B-18}$$

In which

$$g = \frac{7 - 5\nu_m}{2(4 - 5\nu_m)}$$

Note also that (4.13) satisfies the slope conditions (5.7) and (5.9), as it should.

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Table 1

Experimental results for WC-Co alloy (Ref. 17)

c	E(10^6 psi)	contiguity
0.0	30	
0.10	33	0.16
0.35	46 - 48	0.29
0.50	54 - 55	0.42
0.63	61.5	0.47
0.78	72.5	0.54
0.90	88	0.66
1.00	102	

Table 2

Theoretical results for moduli of WC-Co alloy

c	$\frac{K^*}{K_m}$	$\frac{G_1^*}{G_m}$	$\frac{\overline{G}^*}{G_m}$	$\frac{G_2^*}{G_m}$	$\frac{E_1^*}{E_m}$	$\frac{\overline{E}^*}{E_m}$	$\frac{E_2^*}{E_m}$
0.00	1	1	1	1	1	1	1
0.20	1.167	1.227	1.262	1.287	1.218	1.245	1.269
0.40	1.373	1.516	1.600	1.665	1.495	1.561	1.619
0.50	1.495	1.711	1.807	1.887	1.678	1.753	1.823
0.60	1.633	1.953	2.049	2.133	1.904	1.976	2.050
0.80	1.970	2.627	2.678	2.733	2.516	2.549	2.599
0.90	2.181	3.080	3.098	3.120	2.919	2.932	2.951
1.00	2.428	3.622	3.622	3.622	3.400	3.400	3.400

c	ν_1^*	$\overline{\nu}^*$	ν_2^*
0.00	0.30	0.30	0.30
0.20	0.291	0.287	0.282
0.40	0.282	0.273	0.264
0.50	0.275	0.265	0.256
0.60	0.267	0.258	0.249
0.80	0.245	0.241	0.236
0.90	0.232	0.230	0.229
1.00	0.22	0.22	0.22

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